BCA - Ist

9-2-25

Prove that $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$

Proof → $\text{L.H.S.} = \binom{n}{k} + \binom{n}{k-1}$

$$= \frac{\underline{\lfloor n}}{\underline{\lfloor k}\ \underline{\lfloor n-k}} + \frac{\underline{\lfloor n}}{\underline{\lfloor k-1}\ \underline{\lfloor n-k+1}}$$

$$= \frac{\underline{\lfloor n}\ (n-k+1)}{\underline{\lfloor k}\ \underline{\lfloor n-k}\ (n-k+1)} + \frac{\underline{\lfloor n} \cdot k}{\underline{\lfloor k-1}\ k\ \underline{\lfloor n-k+1}}$$

(Multiplying $n-k+1$ to numerator and denominator to first term and multiplying k to the second term).

$$= \frac{\underline{\lfloor n}\ (n-k+1)}{\underline{\lfloor k}\ \underline{\lfloor (n-k+1)}} + \frac{k\ \underline{\lfloor n}}{\underline{\lfloor k-1}\ k\ \underline{\lfloor n-k+1}}$$

$$= \frac{\underline{\lfloor n}\ (n-k+1)}{\underline{\lfloor k}\ \underline{\lfloor (n-k+1)}} + \frac{k\ \underline{\lfloor n}}{\underline{\lfloor k}\ \underline{\lfloor (n-k+1)}} \qquad \left(\text{since } k\ \underline{\lfloor k-1} = \underline{\lfloor k}\right)$$

$$= \frac{\underline{\lfloor n}(n-k+1) + k\ \underline{\lfloor n}}{\underline{\lfloor k}\ \underline{\lfloor (n-k+1)}}$$

$$= \frac{n\ \underline{\lfloor n} - \cancel{k\ \underline{\lfloor n}} + \underline{\lfloor n} + \cancel{k\ \underline{\lfloor n}}}{\underline{\lfloor k}\ \underline{\lfloor (n-k+1)}} = \frac{n\ \underline{\lfloor n} + \underline{\lfloor n}}{\underline{\lfloor k}\ \underline{\lfloor n-k+1}}$$

$$= \frac{\underline{\lfloor n}\ (n+1)}{\underline{\lfloor k}\ \underline{\lfloor n-k+1}} = \frac{\underline{\lfloor n+1}}{\underline{\lfloor k}\ \underline{\lfloor n-k+1}}$$

$$= \frac{\underline{\lfloor n+1}}{\underline{\lfloor k}\ \underline{\lfloor n+1-k}} = \binom{n+1}{k} = \text{R.H.S. } \underline{\underline{\text{Proved}}}$$

Q. Prove that for all integers n and k with $1 \le k \le n$ we have

$$\binom{n}{k-1} + 2\binom{n}{k} + \binom{n}{k+1} = \binom{n+2}{k+1}$$

Solution → By property $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ we have,

$$\binom{n}{k-1} + 2\binom{n}{k} + \binom{n}{k+1} = \binom{n}{k-1} + \binom{n}{k} + \binom{n}{k} + \binom{n}{k+1}$$

$$= \binom{n}{k-1} + \binom{n}{k} + \binom{n}{k} + \binom{n}{k+1}$$

$$= \binom{n+1}{k} + \binom{n}{k} + \binom{n}{k+1}$$ [By the property of Binomial coefficient $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$]

$$= \binom{n+1}{k} + \binom{n+1}{k+1}$$

$$= \binom{n+2}{k+1}$$ Proved (By above property).

Permutations —

Definition — A permutation of a set X is a rearrangement of its elements. Example — Let $X = \{1, 2\}$. Then there are 2 permutations — $\{1, 2\}$, $\{2, 1\}$.

Let $X = \{1, 2, 3\}$. Then there are 6 permutations —

123, 132, 213, 231, 312, 321.

Let $X = \{1, 2, 3, 4\}$. Then there are 24 permutations

Hence, one can show that there are exactly $\underline{|n}$ permutations of the n elements Set X.

A permutation of a set X is a one-one correspondence from X to itself.

Notation — Let $X = \{1, 2, 3, \ldots n\}$ and $\alpha : X \to X$ be a permutation; It is convenient to describe this function in the following way:—

$$\alpha = \begin{pmatrix} 1 & 2 & \cdots\cdots & n \\ \alpha(1) & \alpha(2) & \cdots & \alpha(n) \end{pmatrix}.$$

Definition — Let $X = \{1, 2, 3, 4, \cdots n\}$ and $\alpha : X \to X$ be a permutation. Let $i_1, i_2, i_3 \ldots i_r$ be distinct numbers from $\{1, 2, 3, \cdots n\}$.

If $\alpha(i_1) = i_2$, $\alpha(i_2) = i_3$, ..., $\alpha(i_{r-1}) = i_r$,

$\alpha(i_r) = i_1$, and $\alpha(i_v) = i_v$ for other numbers from $\{1, 2, 3, \ldots, n\}$, then α is called an r-cycle.

An r-cycle is denoted by $(i_1, i_2, i_3, \ldots i_r)$.

Example :- We can use different notations for the same cycle. For example

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = (1) = (2) = (3),$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1\ 2\ 3) = (2\ 3\ 1) = (3\ 1\ 2)$$

Formula is :- The number of permutations of n different things taken r at a time is denoted by $^nP_r = \frac{\lfloor n}{\lfloor n-r}$

$\boxed{\text{Where } \lfloor n = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n}$.

Some important formulas of Permutation :-

(1) If there are n elements of which a_1 are alike to some kind, a_2 are alike to another kind, a_3 are alike to third kind and so on and a_r are of rth kind, where $(a_1 + a_2 + a_3 + \cdots + a_r) = n$

Then, number of permutations of these n objects is $\lfloor n / [\lfloor a_1 \cdot \lfloor a_2 \cdot \lfloor a_3 \cdots \lfloor a_r]$

(2) Number of permutations of n distinct elements taking n elements at a time $^nP_n = \lfloor n$

(3) The number of permutations of n dissimilar elements taking r elements at a time, when x particular things always occupy definite places $= {}^{n-x}P_{r-x}$

(4) The number of permutations of n dissimilar elements when r specified things always come together is $= \lfloor r \, \lfloor n-r+1$

(5) The number of permutations of n dissimilar elements when r specified things never come together is $= \lfloor n - \left[\lfloor r \cdot \lfloor n-r+1 \right]$

(6) The number of circular permutations of n different elements taken x elements at time is $= {}^{n}P_{x} / x$

(7) The number of circular permutations of n different things $= {}^{n}P_{n}/n$.