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Binomial Coefficient :-

Factorial is represented by mathematical terms as :-

$$n! = \lfloor n = 1 \cdot 2 \cdot 3 \cdot 4 \cdots\cdots n$$

or

$$\prod_{j=1}^{n} a_j = a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdots\cdots a_n.$$

$$\boxed{\lfloor 0 = 0! = 1}$$

Factorials are used to define Binomial coefficients.

Definition – Let m and k are two non-negative integers with $k \leq m$, then Binomial coefficient $\binom{m}{k}$ is defined as :—

$$\binom{m}{k} = \frac{m!}{k!\,(m-k)!}$$

Some properties of Binomial Coefficient :-

(1) $\binom{n}{0} = \binom{n}{n} = 1$

(2) $\binom{n}{1} = \binom{n}{n-1} = n$

(3) $\binom{n}{k} = \binom{n}{n-k}$

(4) $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$

The Binomial Theorem — Let a and b be real numbers and n be any non-negative integers, then

$$\boxed{(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \cdots\cdots \cdots + \binom{n}{n-2} a^2 b^{n-2} + \binom{n}{n-1} a b^{n-1} + b^n.}$$

Problem:- For all integer $n \geq 1$ we have

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

Proof:- Putting $a = b = 1$ in the Binomial theorem, we have

$$(1+1)^n = 1^n + \binom{n}{1} \cdot 1^{n-1} \cdot b + \binom{n}{2} \cdot 1^{n-2} \cdot 1^2 + \cdots\cdots$$
$$+ \binom{n}{n-2} \cdot 1^2 \cdot 1^{n-2} + \binom{n}{n-1} \cdot 1 \cdot 1^{n-1} + 1^n.$$

Hence, $2^n = 1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-2} + \binom{n}{n-1} + 1$

Therefore, by property I of Binomial coefficient,

$\binom{n}{0} = \binom{n}{n} = 1$, we have the result —

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-2} + \binom{n}{n-1} + \binom{n}{n}$$

Proved.

Problem:- For all integers $n \geq 1$, we have

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0$$

Proof:- Putting $a = 1$ and $b = -1$ in the Binomial theorem, we get

$$(1-1)^n = 1^n + \binom{n}{1} \cdot 1^{n-1} \cdot (-1) + \binom{n}{2} \cdot 1^{n-2} (-1)^2 + \cdots\cdots$$
$$+ \binom{n}{n-1} \cdot 1 \cdot (-1)^{n-1} + (-1)^n,$$

Hence,

$$0 = 1 - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^{n-1} \binom{n}{n-1} + (-1)^n,$$

Therefore by property I of Binomial coefficient, we get

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^{n-1} \binom{n}{n-1} + (-1)^n \binom{n}{n}$$

Proved