

- Definition:— If $u = f(x, y)$ be a function of x and y . If the sum of the powers of x and y in each term be equal then the function $f(x, y)$ is called homogeneous function in x and y . If the sum of powers of x and y in each term of $f(x, y)$ be 4 then it is called homogeneous function of order 4 and so on.
- In general, homogeneous fn. of n th degree can be written as below

$$f(x, y) = a_0 x^n + a_1 x^{n-1}y + a_2 x^{n-2}y^2 + \dots + a_n y^n$$
- It can also be written in following form

$$f(x, y) = x^n \phi\left(\frac{y}{x}\right), \text{ where } \phi \text{ is a function of } \left(\frac{y}{x}\right)$$
- If $f(x, y)$ is a homogeneous fn. of 2nd order, then

$$f(x, y) = x^2 \phi\left(\frac{y}{x}\right)$$
- If $f(x, y)$ is a homogeneous fn. of n th order, then

$$f(tx, ty) = t^n f(x, y)$$

Euler's Theorem : If $u = f(x, y)$ be a function of x and y of n th degree then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$$

Solution:— Since $u = f(x, y)$ is a homogeneous in x and y of n th order, then we can

Write

87

$$u = x^n f\left(\frac{y}{x}\right) \dots \dots \dots (1)$$

Differentiating (1) partially w.r.t to x , we get

$$\begin{aligned} \frac{\partial u}{\partial x} &= x^n \cdot f'\left(\frac{y}{x}\right) \cdot \left(-\frac{y}{x^2}\right) + n x^{n-1} \cdot f\left(\frac{y}{x}\right) \\ \Rightarrow &= -x^{n-2} \cdot y f'\left(\frac{y}{x}\right) + n x^{n-1} \cdot f\left(\frac{y}{x}\right) \\ \therefore x \frac{\partial u}{\partial x} &= -x^{n-1} \cdot y \cdot f'\left(\frac{y}{x}\right) + n x^n f\left(\frac{y}{x}\right) \dots \dots \dots (2) \end{aligned}$$

Again differentiating (1) partially w.r.t to y , we get

$$\frac{\partial u}{\partial y} = x^n \cdot f'\left(\frac{y}{x}\right) \cdot \frac{1}{x} = x^{n-1} f'\left(\frac{y}{x}\right)$$

$$\therefore y \frac{\partial u}{\partial y} = x^{n-1} \cdot y f'\left(\frac{y}{x}\right) \dots \dots \dots (3)$$

$$\begin{aligned} (2) + (3) \Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} &= -x^{n-1} y f'\left(\frac{y}{x}\right) + n x^n f\left(\frac{y}{x}\right) + \cancel{x^{n-1} y f'\left(\frac{y}{x}\right)} \\ &= n x^n f\left(\frac{y}{x}\right) = n u \quad [\text{from (1)}] \\ \therefore \boxed{x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n u} &- \text{Proved.} \end{aligned}$$

Note:- Also we can show that if $u = f(x, y, z)$ be a homogeneous function of order n , then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = n u$$

Q.(1) If $u = \sin^{-1} \frac{x^2 + y^2}{x+y}$, show that
 $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \tan u$.

38

Solution:- Given that $u = \sin^{-1} \left(\frac{x^2 + y^2}{x+y} \right)$

$$\therefore \sin u = \frac{x^2 + y^2}{x+y} = \frac{x^2 \left(1 + \frac{y^2}{x^2} \right)}{x \left(1 + \frac{y}{x} \right)} = \frac{x \left(1 + \frac{y^2}{x^2} \right)}{1 + \frac{y}{x}}$$

Hence, $\sin u$ is a homogeneous function of order 1.

$$\text{Let } \sin u = v$$

Then v is a function of order 1.

Hence by Euler's theorem

$$x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = 1 \cdot v = v \dots \dots \dots (1)$$

$$\text{Now, } \because v = \sin u$$

$$\therefore \frac{\partial v}{\partial x} = (\cos u) \frac{\partial u}{\partial x} \text{ and } \frac{\partial v}{\partial y} = (\cos u) \frac{\partial u}{\partial y}$$

Then from (1), we get

$$\begin{aligned} \text{L.H.S.} &= x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = x(\cos u) \frac{\partial u}{\partial x} + y(\cos u) \frac{\partial u}{\partial y} \\ &= \cos u \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) \end{aligned}$$

\therefore from using the equation (1), we get

$$\cos u \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) = v = \sin u$$

$$\therefore \boxed{x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \tan u} \quad \text{Proved.}$$

Q. (2) If $u = \tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right)$, prove that

39

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u.$$

Solution : — From the given question, we have

$$\tan u = \frac{x^3 + y^3}{x - y} = \frac{x^3 \left(1 + \frac{y^3}{x^3}\right)}{x \left(1 - \frac{y}{x}\right)} = \frac{x^2 \left(1 + \frac{y^3}{x^3}\right)}{\left(1 - \frac{y}{x}\right)}$$

\therefore tanu is a homogeneous of order 2.

$$\text{Let } \tan u = v$$

Let $\tan u = v$
 Then v will also be a fn. of order 2.

$$\text{Now, } \frac{\partial u}{\partial x} = (\sec^2 u) \frac{\partial u}{\partial x} \text{ and } \frac{\partial u}{\partial y} = (\sec^2 u) \frac{\partial u}{\partial y}$$

From (1), we get

$$\text{from (1), we get } x \cdot (\sec^2 u) \frac{\partial u}{\partial x} + y (\sec^2 u) \frac{\partial u}{\partial y} = 2u = 2 \tan u$$

$$\Rightarrow \sec^2 u \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) = 2 \tan u$$

$$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2 \tan u}{\sec^2 u} = \frac{2 \sin u}{\cos u} \times \cos^2 u$$

$$= 2 \sin u \cos u = \sin 2u$$

$$\therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u \quad \text{Proved.}$$

By

Dr. Bikram Singh

Associate Professor

Dept. of Math

Sher Shah College, Sardarni.

23. 09. 2020.