

[Differential Equation of the First order but not of the First Degree]

- In this chapter we shall consider diff. eqn. which contains the higher powers of $\frac{dy}{dx}$ than one. we shall usually denote $\frac{dy}{dx}$ by p .

The most general form of a diff. eqn. of the first order and n th degree is

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \dots + P_n = 0 \dots \dots \dots (1)$$

where p stand for $\frac{dy}{dx}$ and P_1, P_2, \dots, P_n are functions of x and y .

Such equations may be possible to solve by one or more of the four methods given below. In each case the problem is reduced to solve one or more equations of first order and first degree.

- (i) Equations solvable for p
- (ii) Solvable for y
- (iii) Solvable for x
- (iv) Clairaut's Equations.

Case (i) : — Equations solvable for p : —

Let the equation of n th degree

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \dots + P_n = 0$$

Can be put in the form

$$(p-f_1)(p-f_2)(p-f_3) \dots (p-f_n) = 0$$

In above equation we see that n th degree diff. eqns has n factors, and each factor is of the first order and first degree.

and where $f_1, f_2, f_3, \dots, f_n$ are fns. of x and y .

Hence, there will be n solutions like as

$$\phi_1(x, y, c_1) = 0, \phi_2(x, y, c_2) = 0, \dots, \phi_n(x, y, c_n) = 0$$

Then the general solution of (1) be

$$\phi_1(x, y, c) \phi_2(x, y, c) \dots \phi_n(x, y, c) = 0$$

There is no loss of generality if we take

$$c_1 = c_2 = \dots = c_n \text{ (say)}$$

• Case - (i) Equations solvable for $p (= \frac{dy}{dx})$

Q. (2) Solve: $p^2 - p(e^x + e^{-x}) + 1 = 0$ (OR) $p^2 - 2p \cosh x + 1 = 0$

Solution:— The given equation gives

$$p^2 - pe^x - pe^{-x} + 1 = 0$$

$$\Rightarrow p^2 - pe^x - pe^{-x} + e^x \cdot e^{-x} = 0$$

$$\Rightarrow p(p - e^x) - e^{-x}(p - e^x) = 0$$

$$\text{i.e. } (p - e^x)(p - e^{-x}) = 0$$

Thus, Either $p - e^x = 0$ or $p - e^{-x} = 0$

When $p - e^x = 0$ then $p = e^x$

$$\Rightarrow \frac{dy}{dx} = e^x, \text{ Integrating, we get}$$

$$y = e^x + c \Rightarrow c = y - e^x \dots \dots \dots (1)$$

Again when $p - e^{-x} = 0$ then $p = e^{-x}$

$$\Rightarrow \frac{dy}{dx} = e^{-x}, \text{ Integrating } y = -e^{-x} + c$$

$$\Rightarrow c = y + e^{-x} \dots \dots \dots (2)$$

Equations (1) and (2) give the complete solution

which is $\boxed{(y - e^x - c)(y + e^{-x} - c) = 0}$ Answer.

Q. (2) Solve: - $\left(\frac{dy}{dx}\right)^2 + 2y \cot x \frac{dy}{dx} = y^2$

Solution: - The given equation can be written as

$$p^2 + 2yp \cot x - y^2 = 0$$

$$\Rightarrow p^2 + 2yp \cot x + (y \cot x)^2 = y^2 + y^2 \cot^2 x$$

$$\Rightarrow (p + y \cot x)^2 = y^2 (1 + \cot^2 x) = y^2 \operatorname{cosec}^2 x$$

$$\Rightarrow p + y \cot x = \pm y \operatorname{cosec} x$$

$$\Rightarrow p = -y \cot x \pm y \operatorname{cosec} x$$

$$\therefore \frac{dy}{dx} = -y \cot x + y \operatorname{cosec} x = y (-\cot x + \operatorname{cosec} x) \quad (1)$$

and also $\frac{dy}{dx} = y (-\cot x - \operatorname{cosec} x) \quad \text{--- (2)}$

$$\Rightarrow \frac{dy}{y} = (-\cot x - \operatorname{cosec} x) dx \quad (2)$$

Integrating (1), we get

$$\log y = -\log \sin x + \log \tan \frac{x}{2} + \log c$$

$$\begin{aligned} \Rightarrow \log\left(\frac{y}{c}\right) &= \log\left(\frac{\tan \frac{x}{2}}{\sin x}\right) = \log\left(\frac{\sin \frac{x}{2}}{\cos \frac{x}{2} \cdot 2 \sin \frac{x}{2} \cdot \cos \frac{x}{2}}\right) \\ &= \log\left(\frac{1}{2 \cos^2 \frac{x}{2}}\right) = \log\left(\frac{1}{1 + \cos x}\right) \end{aligned}$$

$$\therefore y = \frac{c}{1 + \cos x}$$

Similarly, integrating (2), we get

$$y = \frac{c}{1 - \cos x}$$

Hence, the general solution of the given eqn. is

$$\left(y - \frac{c}{1 + \cos x}\right) \left(y - \frac{c}{1 - \cos x}\right) = 0$$

Q. (3) Solve $x^2 \left(\frac{dy}{dx}\right)^2 + 3xy \frac{dy}{dx} + 2y^2 = 0$

Solution: - The given equation can be put as

$$x^2 p^2 + 3xy p + 2y^2 = 0$$

$$\Rightarrow x^2 p^2 + 2xy p + xy p + 2y^2 = 0$$

$$\Rightarrow xp(xp + 2y) + y(xp + y) = 0$$

$$\text{i.e. } (xp + y)(xp + 2y) = 0$$

Either $xp + y = 0$ or $xp + 2y = 0$

$$\text{When } xp + y = 0 \Rightarrow x \frac{dy}{dx} = -y \Rightarrow \frac{dx}{x} = -\frac{dy}{y}$$

$$\Rightarrow \frac{dx}{x} + \frac{dy}{y} = 0, \text{ Integrating}$$

$$\log x + \log y = \log c \quad \therefore xy = c \quad \text{--- (1)}$$

Again, when $xp + 2y = 0$ then $x \frac{dy}{dx} + 2y = 0$

$$\Rightarrow x \frac{dy}{dx} = -2y \Rightarrow \frac{dx}{x} = -\frac{1}{2} \frac{dy}{y} \Rightarrow \frac{dx}{x} + \frac{1}{2} \frac{dy}{y} = 0$$

$$\text{Integrating, } \log x + \frac{1}{2} \log y = \log k$$

$$\Rightarrow 2 \log x + \log y = \log k^2 = \log c \quad \left[\because \text{we can write } k^2 = c \right]$$

$$\therefore x^2 y = c \quad \text{--- (2)}$$

Hence equations (1) and (2) constitute the complete solution of the given equation. i.e.

$$\boxed{(xy - c)(x^2 y - c) = 0}$$

By

Dr. Birkarima Singh
Associate Professor
Deptt. of Maths
Sherohak College, Sabaram.
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