

Definition:— An equation containing one or more partial differential coefficients is called partial differential equation (p.d.e.). In this case any dependent variable is a function of two or more independent variables.

If dependent variable  $z$  is a fns. of two variables  $x$  and  $y$  then, we write

$$z = f(x, y).$$

we shall denote the partial derivatives as

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \cdot \partial y} = \frac{\partial^2 z}{\partial y \cdot \partial x} = s$$

$$\text{and } \frac{\partial^2 z}{\partial y^2} = t$$

In this chapter we shall study the origin of p.d.e. and classification of integrals of p.d.e. of first order as made by Lagrange in 1769 and Lagrange's method of finding integral of linear first order partial diff. equation.

• The p.d. eqns. may be obtained in the following two ways—

- (i) Elimination of arbitrary constants.
- (ii) Elimination of arbitrary functions.

• (i) — Let a function  $z$  of  $x, y$  be such that

$$f(x, y, z, a, b) = 0 \dots \dots \dots (1)$$

Differentiating it partially w.r.t.  $x$  and  $y$  and eliminating the constants  $a$  and  $b$  from (1)

then p.d.e. is obtained.

Note: We observe that if the number of constants is equal to the no. of independent variables then the derived p.d.e. is of first order.

But if the no. of consts is greater than the no. of independent variables then the derived p.d.e. will be of the second or higher order.

Q.1) Find the p.d.e. by eliminating  $a$  and  $b$  from

$$z = ax + by + a^2 + b^2$$

Solution: - The given equation is

$$z = ax + by + a^2 + b^2 \dots \dots \dots (1)$$

Differentiating (1) partially w.r. to  $x$  and  $y$ , we get

$$\frac{\partial z}{\partial x} = a \quad \text{and} \quad \frac{\partial z}{\partial y} = b$$

Substituting these values of  $a$  and  $b$  in (1), we get

$$z = x \cdot \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

$$\text{i.e. } \boxed{z = xp + yq + p^2 + q^2}$$

which is the required partial diff. equation.

Q.2) Find p.d.e. by eliminating  $h$  and  $k$  from

$$(x-h)^2 + (y-k)^2 + z^2 = \lambda^2$$

Solution: - Given that

$$(x-h)^2 + (y-k)^2 + z^2 = \lambda^2 \dots \dots \dots (1)$$

Differentiating (1) partially w.r. to  $x$  and  $y$ , we get

$$2(x-h) + 2z \frac{\partial z}{\partial x} = 0 \Rightarrow (x-h) = -z \frac{\partial z}{\partial x} \dots (2)$$

$$\text{and } 2(y-k) + 2z \frac{\partial z}{\partial y} = 0 \Rightarrow (y-k) = -z \frac{\partial z}{\partial y} \dots (3)$$

From (1), (2) and (3), we get

$$z^2 \left( \frac{\partial z}{\partial x} \right)^2 + z^2 \left( \frac{\partial z}{\partial y} \right)^2 + z^2 = \lambda^2$$

i.e.  $z^2 \left[ \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 + 1 \right] = \lambda^2$

or,  $\boxed{z^2 (p^2 + q^2 + 1) = \lambda^2}$  - which is the required equation.

Q(3) - Find the p.d.e. by eliminating a, b, c

from  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Solution:- Given equation be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \dots\dots (1)$$

Differentiating (1) partially w.r.t. x and y, we get

$$\frac{2x}{a^2} + \frac{2z}{c^2} \cdot \frac{\partial z}{\partial x} = 0 \Rightarrow c^2 x + a^2 z \cdot \frac{\partial z}{\partial x} = 0 \dots\dots (2)$$

and  $\frac{2y}{b^2} + \frac{2z}{c^2} \cdot \frac{\partial z}{\partial y} = 0 \Rightarrow c^2 y + b^2 z \cdot \frac{\partial z}{\partial y} = 0 \dots\dots (3)$

Again diff. (2) and (3) partially, we get

$$c^2 + a^2 \left[ z \cdot \frac{\partial^2 z}{\partial x^2} + \left( \frac{\partial z}{\partial x} \right)^2 \right] = 0$$

$$\Rightarrow c^2 + a^2 \left( \frac{\partial z}{\partial x} \right)^2 + a^2 z \frac{\partial^2 z}{\partial x^2} = 0 \dots\dots (4)$$

and similarly

$$c^2 + b^2 \left( \frac{\partial z}{\partial y} \right)^2 + a^2 z \frac{\partial^2 z}{\partial y^2} = 0 \dots\dots (5)$$

From (4) and (5) we have (2), we have

$$c^2 = - \frac{a^2 z}{x} \cdot \frac{\partial z}{\partial x}$$

Putty this value in (4), we get

$$- \frac{a^2 z}{x} \cdot \frac{\partial z}{\partial x} + a^2 \left( \frac{\partial z}{\partial x} \right)^2 + a^2 z \frac{\partial^2 z}{\partial x^2} = 0$$

$$\Rightarrow -\frac{z}{x} \cdot \frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2 + z \frac{\partial^2 z}{\partial x^2} = 0$$

$$\Rightarrow z x \frac{\partial^2 z}{\partial x^2} + x \left(\frac{\partial z}{\partial x}\right)^2 - z \cdot \left(\frac{\partial z}{\partial x}\right) = 0$$

$$\text{i.e. } xzr + xp^2 - zp = 0 \dots \dots \dots (6)$$

which is the required solution.

and other solutions are also derived easily.

### (ii) Elimination of Arbitrary Functions:

Let the given equation be

$$F(u, v) = 0 \dots \dots \dots (1)$$

where  $u$  and  $v$  are fns. of  $x, y, z$ .

where  $z$  be dependent and  $x$  and  $y$  be independent variables so that

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial y}{\partial x} = 0, \quad \frac{\partial x}{\partial y} = 0$$

Differentiating (1) partially w.r.t.  $x$ , we obtain

$$\frac{\partial F}{\partial u} \left[ \frac{\partial u}{\partial x} \cdot \frac{\partial z}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial x} \right] +$$

$$\frac{\partial F}{\partial v} \left[ \frac{\partial v}{\partial x} \cdot \frac{\partial z}{\partial x} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial x} + \frac{\partial v}{\partial z} \cdot \frac{\partial z}{\partial x} \right] = 0$$

$$\Rightarrow \frac{\partial F}{\partial u} \left[ \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} \cdot p \right] + \frac{\partial F}{\partial v} \left[ \frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} \cdot p \right] = 0$$

$$\therefore \frac{\partial F}{\partial u} / \frac{\partial F}{\partial v} = - \frac{\frac{\partial u}{\partial x} + p \frac{\partial u}{\partial z}}{\frac{\partial v}{\partial x} + p \frac{\partial v}{\partial z}} \dots \dots \dots (2)$$

Similarly, differentiating (1) w.r.t.  $y$ , we obtain

$$\frac{\partial F}{\partial u} / \frac{\partial F}{\partial v} = \frac{\frac{\partial u}{\partial y} + q \frac{\partial u}{\partial z}}{-\frac{\partial v}{\partial y} + q \frac{\partial v}{\partial z}} \dots \dots \dots (3)$$

Eliminating  $\frac{\partial F}{\partial u}$  &  $\frac{\partial F}{\partial v}$  from (2) and (3)

we have

$$\left(\frac{\partial u}{\partial x} + p \frac{\partial u}{\partial z}\right) \left(\frac{\partial u}{\partial y} + q \frac{\partial u}{\partial z}\right) = \left(\frac{\partial u}{\partial x} + p \frac{\partial u}{\partial z}\right) \left(\frac{\partial v}{\partial y} + q \frac{\partial v}{\partial z}\right)$$

which can be written as

$$\left(\frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial z} - \frac{\partial u}{\partial y} \cdot \frac{\partial u}{\partial z}\right) p + \left(\frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial z} - \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial z}\right) q = \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial z} \dots \dots \dots (4)$$

$$\Rightarrow p p + q q = R \dots \dots \dots (5)$$

where,  $p = \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial z} - \frac{\partial u}{\partial y} \cdot \frac{\partial u}{\partial z} = \frac{\partial(u, u)}{\partial(x, z)}$

$q = \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial z} - \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial z} = \frac{\partial(u, u)}{\partial(z, x)}$

and  $R = \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial u}{\partial x} = \frac{\partial(u, u)}{\partial(x, y)}$

The equation  $pp + qq = R$  is a linear p.d.e. of first order and first degree in  $p$  and  $q$ .

Q.1 Find the p.d.e. form

$$\phi(x+y+z, x^2+y^2-z^2) = 0$$

Solution:- Let  $x+y+z = u$  and  $x^2+y^2-z^2 = v$   
 so that  $\phi(u, v) = 0 \dots \dots \dots (1)$

Differentiating (1) partially w.r. to  $x$ , we get

$$\frac{\partial \phi}{\partial u} \left[ \frac{\partial u}{\partial x} + p \frac{\partial u}{\partial z} \right] + \frac{\partial \phi}{\partial v} \left( \frac{\partial v}{\partial x} + p \frac{\partial v}{\partial z} \right) = 0$$

i.e.  $\frac{\partial \phi}{\partial u} (1+p) + \frac{\partial \phi}{\partial v} (2x - 2pz) = 0 \dots \dots \dots (2)$

Again differentiating (1) partially w.r.t.  $y$ ,  
we get

$$\frac{\partial \phi}{\partial u} \left[ \frac{\partial u}{\partial y} + q \frac{\partial u}{\partial z} \right] + \frac{\partial \phi}{\partial v} \left( \frac{\partial u}{\partial y} + q \frac{\partial u}{\partial z} \right) = 0$$

i.e.  $\frac{\partial \phi}{\partial u} (1+q) + \frac{\partial \phi}{\partial v} (2y - 2zq) = 0 \dots (3)$

Eliminating  $\frac{\partial \phi}{\partial u}$  and  $\frac{\partial \phi}{\partial v}$ , we get

$$(1+p)(2y - 2zq) - (1+q)(2x - 2zp) = 0$$

i.e.  $\boxed{(y+z)p - (x+z)q = x-y}$

Note: • A differential equation involving partial derivatives  $p$  and  $q$  only and no higher order is called order one.

• If the degree (or power) of  $p$  and  $q$  is unity then it is linear p.d.e. of order one.

[Equation  $p + q = R$  is the standard form of the linear p.d.e. of order one.]

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07. 07. 2020