

Problems solution on total differential Equations.

49.

- Method-2: One variable regarded as constant. If the condition of integrability is satisfied and any two terms can easily be solved then the third variable z (say) is taken as constant so that

$$dz = 0.$$

The following examples will make the method clear.

Ex. (2) solve $3x^2 dx + 3y^2 dy - (x^3 + y^3 + e^{2z}) dz = 0$

Solution:— Here, $P = 3x^2$, $Q = 3y^2$ and $R = -(x^3 + y^3 + e^{2z})$.
Now, we can easily see that the condition of integrability is satisfied.

Here, we take $z = \text{Constant}$ so that $dz = 0$

∴ The given equation becomes

$$3x^2 dx + 3y^2 dy = 0 \dots \dots \dots (1)$$

Integrating,

$$x^3 + y^3 = \phi(z) \text{ (say)} \dots \dots \dots (2)$$

where $\phi(z)$ is a function of z only (regarded const.).

Now, differentiating (2), we have

$$3x^2 dx + 3y^2 dy - \phi'(z) dz = 0 \dots \dots \dots (3)$$

Comparing (3) with the given equation, we get

$$\phi'(z) = x^3 + y^3 + e^{2z} = \phi(z) + e^{2z} \text{ [from (2)]}$$

i.e. $\frac{d\phi}{dz} - \phi = e^{2z}$ which is linear.

$$I.F. = e^{\int -1 dz} = e^{-z}$$

$$\therefore \phi e^{-z} = \int e^{2z} \cdot e^{-z} dz + C = \int e^z dz + C = e^z + C$$

$$\Rightarrow \phi = e^{2x} + ce^x$$

$$\text{i.e. } \boxed{x^3 + y^3 = e^{2x} + ce^x}$$

which is required solution.

Q. (2) Solve: $(2x^2 + 2xy + 2xz^2 + 1)dx + dy + 2zdz = 0$

Solution: It can be easily seen that the condition of integrability is satisfied.

Now, we take $x = \text{Constant}$ so that $dx = 0$
Then the given eqn becomes

$$dy + 2zdz = 0$$

Integrating, $y + z^2 = \phi(x) \dots \dots \dots (1)$

where $\phi(x)$ is a fn. of x only.

Differentiating (1), we get

$$dy + 2zdz = \phi'(x)dx$$

$$\Rightarrow \phi'(x)dx + dy + 2zdz = 0 \dots \dots \dots (2)$$

Comparing (2) with the given equation

$$-\phi'(x) = (2x^2 + 2xy + 2xz^2 + 1)$$

$$\Rightarrow -\phi'(x) = 2x^2 + 2x(y + z^2) + 1 = 2x^2 + 2x\phi(x) + 1$$

$$\Rightarrow \frac{d\phi}{dx} + (2x)\phi = -(2x^2 + 1)$$

which is linear equation.

$$\text{I.F.} = e^{\int 2x dx} = e^{x^2}$$

$$\therefore \phi e^{x^2} = -\int (2x^2 + 1) \cdot e^{x^2} dx + C$$

$$= -\int x(2x \cdot e^{x^2}) dx - \int e^{x^2} dx + C$$

$$= -x e^{x^2} + \int 1 \cdot e^{x^2} dx - \int e^{x^2} dx + C$$

$$= -x e^{x^2} + C$$

Integrating we have

$$\Rightarrow \phi = -x + C e^{-x^2}$$

$$\Rightarrow y + z^2 + x = C e^{-x^2}$$

$\therefore C = (y + x + z^2) e^{x^2}$ is required solution.

- Method-(3): If P, Q, R are homogeneous functions of x, y, z then one variable say, z may be separated from the other two by the substitution $x = zu$ and $y = zv$

so that $dx = z du + u dz$ and $dy = z dv + v dz$

then the given equation will be reduced to the form in which either the coefficient of dz is zero or not zero. In both cases the new equation may easily be integrated.

Q. (1) Solve: $(yz + z^2) dx - xz dy + xy dz = 0$

Solution: First of all, we shall show that the condition of integrability is satisfied.

Here, P, Q and R are homogeneous fns.

Now, putting $x = uz$ and $y = vz$

so that $dx = z du + u dz$ and $dy = v dz + z dv$

Putting these values of dx and dy in the given eqn. we find that

$$(yz + z^2)(z du + u dz) - xz(v dz + z dv) + (uz)(vz) dz = 0$$

$$\Rightarrow \underbrace{uz^2 du + z^3 du + yz u dz + z^2 u dz - xz v dz - xz^2 dv + uvz^2 dz = 0}_{\Rightarrow \frac{z^2}{y} du +}$$

$$(uz^2 + z^2)(z du + u dz) - uz^2(z du + u dz) + uuz^2 dz = 0$$

$$\Rightarrow z[(u+1)du - u du] + [u(u+1) - u^2 + u^2] dz = 0$$

$$\Rightarrow \frac{(u+1)du - u du}{u(u+1)} + \frac{dz}{z} = 0$$

$$\Rightarrow \frac{du}{u} - \frac{du}{u+1} + \frac{dz}{z} = 0$$

Integrating,

$$\log u - \log(u+1) + \log z = \log c$$

$$\Rightarrow \log\left(\frac{uz}{u+1}\right) = \log c \Rightarrow uz = c(u+1)$$

$$\Rightarrow z = \frac{c}{z}(uz+z) \Rightarrow \boxed{xz = c(y+z)}$$

Which is the required condition.

Q(2) Solve

$$(x^2y - y^3 - y^2z)dx + (xy^2 - x^2z - x^3)dy + (xy^2 + x^2y)dz = 0$$

Solution:- It can be seen easily that the condition of integrability is satisfied.

It can also be solved by inspection method

Dividing the given equation by x^2y^2 , we have

$$\left(\frac{1}{y} - \frac{y}{x^2} - \frac{z}{x^2}\right)dx + \left(\frac{1}{x} - \frac{z}{y^2} - \frac{x}{y^2}\right)dy + \left(\frac{1}{x} + \frac{1}{y}\right)dz = 0$$

$$\Rightarrow \left[\frac{(x^2 - y - z)dx}{x^2y} + \frac{(y^2 - xz - x)dy}{x^2y} + \frac{(y+x)dz}{xy} \right] = 0$$

$$\Rightarrow \frac{x dz - z dx}{x^2} + \frac{x dy - y dx}{x^2} + \frac{y dx - x dy}{y^2}$$

$$+ \frac{y dz - z dy}{y^2} = 0$$

$$= -x e^a + C$$

Integrating, we have

$$\frac{z}{x} + \frac{y}{x} + \frac{x}{y} + \frac{z}{y} = C$$

$$\text{i.e. } x^2 + y^2 + z(x+y) = Cxy$$

which is required solution.

By Method - 3 (Homogeneous method)

Here P, Q, R are homogeneous functions in x, y, z.

Putting $x = zu$ and $y = zv$ so that

$$dx = zdu + u dz, \quad dy = zdv + v dz$$

Putting these values of dx and dy in the given eqn.

we get

$$(z^3 u^2 v - z^3 v^3 - z^3 v^2)(zdu + u dz) + (z^3 u v^2 - z^3 u^2 - z^3 u^3)$$

$$(zdv + v dz) + (z^3 u v^2 + z^3 u^2 v) dz = 0$$

$$\Rightarrow [(u^2 v - v^3 - v^2) du + (u v^2 - u^2 - u^3) dv] dz$$

$$+ [(u^2 v - v^3 - v^2) u + (u v^2 - u^2 - u^3) v + (u v^2 + u^2 v) dz] = 0$$

$$\Rightarrow (u^2 v - v^3 - v^2) du + (u v^2 - u^2 - u^3) dv = 0$$

Dividing by $u^2 v^2$, we have

$$\left(\frac{1}{v} - \frac{v}{u^2} - \frac{1}{u^2}\right) du + \left(\frac{1}{u} - \frac{1}{v^3} - \frac{1}{v^2}\right) dv = 0$$

$$\Rightarrow \frac{-v du + u dv}{u^2} + \frac{v du - u dv}{v^2} - \frac{1}{u^2} du - \frac{1}{v^2} dv = 0$$

$$\text{Integrating, } \frac{v}{u} + \frac{u}{v} + \frac{1}{u} + \frac{1}{v} = C$$

$$\Rightarrow v^2 + u^2 + u + v = Cuv \quad \text{or } (zu)^2 + (zv)^2 + z^2 u + z^2 v = C z^2 uv$$

$$\text{i.e. } \boxed{y^2 + x^2 + z(y+x) = Cxy}$$

which is required solution

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